JOURNAL OF COMPUTATIONAL PHYSICS 3, 473–485 (1969)

Stabilization of Difference Schemes for the Equations of Inviscid Compressible Flow by Artificial Diffusion

B. VAN LEER

University Observatory, Leiden, Netherlands

Received August 9, 1968

ABSTRACT

In this paper we present a stability analysis of the general diffusion-stabilized difference scheme for the equations of inviscid compressible flow in conservation form. Both one-step (first-order) and two-step (second-order) methods are examined, for an arbitrary number of space dimensions. All schemes considered are found to be governed by the same stability condition on the local diffusion coefficient. The lower limit of this coefficient is proven proportional to the *square* of the local maximum characteristic speed; the best current method (Rusanov's) employs only *linear* dependence.

1. INTRODUCTION

During a search for numerical methods applicable to problems of interstellar gas dynamics we examined those difference schemes for the equations of inviscid compressible flow (ICF) that owe their stability to the inclusion of an artificial diffusion term (for an example, see Lax [1]). A linear stability analysis of the *general* diffusion-type scheme, as presented below, reveals a class of optimal schemes (incorporating the *minimum* diffusion required for stability) that may fill the gap in accuracy between the best-performing first-order technique, *viz.* Rusanov's [2], and the second-order technique due to Lax and Wendroff [3].

Most of our derivations apply to an arbitrary hyperbolic system of conservation laws (HSCL) in one or two space dimensions. This system may be thought to represent the Lagrangean and/or Eulerian equations of ICF in one or two Cartesian coordinates (see refs. [1], [3]). The use of spherical or cylindrical coordinates introduces nondifferentiated terms into the equations; this however does not affect a stability analysis like ours, which is based on the Von Neumann criterion (for a discussion see Richtmyer and Morton's book [4], Section 4.7).

2. Preliminary Considerations

A HSCL in one space dimension may be written as

$$w_t + f_x = 0 \tag{1}$$

where the vector w denotes a number (say p) of functions of x and t, and f is a vector of the same dimension, whose components are functions of w only.[1] As mentioned before, such a system may stand for the Lagrangean or Eulerian equations of ICF in the case of slab symmetry. It is often convenient to write (1) as

$$w_t + Aw_x = 0, \tag{2}$$

the matrix A being the Jacobian of f with respect to w and generally depending on w. To make the system hyperbolic we require that A must have p real and distinct eigenvalues. For ICF these represent the characteristic speeds, as is assumed to be understood by the reader. We call the eigenvalues $a^{(k)}(k = 1,..., p)$ and define

$$a \equiv \operatorname*{Max}_k | a^{(k)} |, \tag{3}$$

a parameter often needed in the course of a stability analysis. For a HSCL in two space dimensions we write

$$w_t + f_x + h_y = 0 \tag{4}$$

or

$$w_t + Aw_x + Bw_y = 0, \tag{5}$$

the matrix B being the Jacobian of h with respect to w. Eq. (4) will represent the Eulerian equations of plane ICF.

Difference schemes will be operating in a spatial mesh built up of discrete coordinates x_m and, possibly, y_n, respectively spaced by the constants Δx and Δy. At a given instant t^j the size of the next time step Δt^j is restricted by a certain stability condition, *viz.* the Courant–Friedrichs–Lewy (CFL) condition. Henceforth a discrete value $q(t^j, x_m, y_n)$ of any function q may be abbreviated to q^j_{mn}; in the one-dimensional case the second subscript is of course omitted.

When constructing a difference scheme for eq. (1) or (4) we start from the following common expansion

$$w(t + \Delta t) = w(t) + \Delta t\, w_t + \frac{(\Delta t)^2}{2} w_{tt} + \frac{(\Delta t)^3}{6} w_{ttt} + \cdots. \tag{6}$$

Because we are dealing with an initial value problem it is possible to transform all

[1] This is not the most general form of a HSCL as defined in ref. [1] but the form considered in ref. [3].

time derivatives into space derivatives with aid of the governing differential equation. For instance, in the one-dimensional case eq. (6) passes into

$$w(t + \Delta t) = w(t) - \Delta t\, A w_x + \frac{(\Delta t)^2}{2} (A^2 w_x)_x - \frac{(\Delta t)^3}{6} (A^3 w_x)_{xx} + \cdots. \tag{7}$$

The order of accuracy of a difference scheme is designated by the number of *properly evaluated* time derivatives it includes. We stress herewith the fact that the errors involved in the replacement of spatial derivatives by difference quotients should not be of a lower order than the overall truncation error (i.e. the truncated part of the expansion (6)).

To examine the stability of a difference scheme based e.g. on eq. (5) we let it act on a smooth set of initial values w superimposed by some small complex perturbation w' given by

$$w'^{j}_{mn} = \epsilon w^{j}_{00} \exp[2\pi i(x_m/l_1 + y_n/l_2)], \qquad \epsilon \ll 1. \tag{8}$$

Clearly l_1 and l_2 are the wavelengths of certain oscillations in the x- and y-direction; the frequently occurring expressions $2\pi \Delta x/l_1$ and $2\pi \Delta y/l_2$ are usually replaced by α and β. In the one-dimensional case the dependence on y is left out of eq. (8). The local amplification matrix G is now defined by

$$w'^{j+1}_{mn} = \{G + O(\Delta x)\}\, w'^{j}_{mn} \quad \text{for} \quad \epsilon \to 0. \tag{9}$$

As indicated by the above definition, terms in G of the order $O(\Delta x)$ do not matter and therefore will always be dropped. In practice this is equivalent to regarding w, or any function of w, x and y, as a constant in the vicinity of the mesh point (x_m, y_n).

The given oscillations will not grow in amplitude provided that the eigenvalues $g^{(k)} (k = 1, \ldots, p)$ of G do not exceed one in absolute value. Hence, if

$$g \equiv \operatorname*{Max}_{k} | g^{(k)} | \tag{10}$$

a necessary condition for stability becomes

$$g \leqslant 1 \qquad \text{for any } \alpha \text{ and } \beta. \tag{11}$$

Sufficiency of this condition cannot be warranted because it is merely based on a linear analysis (involving only an infinitesimal variation of w); eventually numerical experiments must prove its validity.

3. ONE-STEP SCHEMES

3.1. *One dimensional case*

The simplest centered[2] difference approximation of eq. (1) is

$$w_m^{j+1} = w_m^j - \Delta t^j \frac{f_{m+1}^j - f_{m-1}^j}{2\Delta x} \tag{12}$$

with truncation error $O(\Delta t^2)$. Its amplification matrix is given by

$$G_0 = I - i\lambda^j A \sin \alpha, \tag{13}$$

where

$$\lambda^j = \frac{\Delta t^j}{\Delta x}, \tag{14}$$

a *zero-order* parameter. The superscripts of Δt and λ are usually omitted. From eq. (13) and the definitions (3) and (10) it follows that

$$g_0^2 = 1 + \lambda^2 a^2 \sin^2 \alpha, \tag{15}$$

which means that scheme (12) is unconditionally unstable.

The only way to cure it is a modification reducing the real part of G_0. This can be accomplished by adding to the right-hand part of (12) a term of order $O(\Delta x^2)$ comprising a *second derivative.* Lax and Wendroff [3] insert a difference approximation of the *correct* second-order term

$$\frac{(\Delta t)^2}{2} (A^2 w_x)_x, \tag{16}$$

thus reducing the truncation error to $O(\Delta t^3)$. In view of the nonlinear instabilities inherent in their method (see e.g. Burstein [5]) we prefer to employ a simpler expression—not involving the matrix A—of the following general form

$$\frac{(\Delta x)^2}{2} (\kappa w_x)_x \tag{17}$$

which will be interpreted as a *diffusion* term. The diffusion coefficient κ may be any positive *scalar* function (in the mathematical sense) of t and x, and/or w. The factor κ has been brought inside the outer differentation because diffusion terms of the form

$$\frac{(\Delta x)^2}{2} \kappa w_{xx} \tag{18}$$

[2] The centering refers only to difference approximations of spatial derivatives; it is carried on throughout this paper in order to avoid implicit diffusion terms as much as possible.

do in general not yield *weak solutions* of eq. (1) (but e.g. shocks travelling at anomalous speeds, as the author experienced).

The centered difference scheme including an approximation of (17) becomes

$$w_m^{j+1} = w_m^j - \frac{\lambda}{2}(f_{m+1}^j - f_{m-1}^j) + \frac{1}{2}\{\kappa_{m+\frac{1}{2}}^j(w_{m+1}^j - w_m^j) - \kappa_{m-\frac{1}{2}}^j(w_m^j - w_{m-1}^j)\} \tag{19}$$

with

$$\kappa_{m\pm\frac{1}{2}}^j \equiv \frac{1}{2}(\kappa_m^j + \kappa_{m\pm1}^j). \tag{20}$$

Eq. (19) may also be regarded as an approximation of the diffusion equation

$$w_t + f_x = \frac{\Delta t}{2\lambda^2}(\kappa w_x)_x . \tag{21}$$

The amplification matrix associated with scheme (19) is given by

$$G_{11} = \{1 - \kappa(1 - \cos\alpha)\}I - i\lambda A \sin\alpha, \tag{22}$$

where κ is now regarded to be locally constant. The indices 1 respectively refer to a *one*-step scheme (or *first*-order accuracy) and *one* space dimension. The maximum factor of growth of G_{11} is determined by the relation

$$g_{11}^2 = 1 - 4\left\{\kappa - \lambda^2 a^2 - (\kappa^2 - \lambda^2 a^2)\sin^2\frac{\alpha}{2}\right\}\sin^2\frac{\alpha}{2}. \tag{23}$$

It is easily seen that the expression between curly brackets will not become negative if and only if

$$\lambda^2 a^2 \leqslant \kappa \leqslant 1. \tag{24}$$

Hence stability is expected provided that the above holds at any time t^j in any point x_m for the adopted value of κ_m^j. Calling

$$\sigma_m^j = \lambda^j a_m^j \tag{25}$$

we may rewrite (24) as

$$(\sigma_m^j)^2 \leqslant \kappa_m^j \leqslant 1 \tag{26}$$

and, recognizing the Courant number

$$\sigma^j \equiv \operatorname*{Max}_m \sigma_m^j , \tag{27}$$

we see that the inequalities (26) also imply the usual Courant–Friedrichs–Lewy condition

$$\sigma^j \leqslant 1. \tag{28}$$

When considering the interval to which the value of κ is restricted, we observe that the *upper* bound (*viz.* $\kappa_m^j \equiv 1$) yields Lax's original scheme

$$w_m^{j+1} = \frac{1}{2}(w_{m-1}^j + w_{m+1}^j) - \frac{\lambda}{2}(f_{m+1}^j - f_{m-1}^j). \tag{29}$$

Inserting the *lower* bound into (17) we obtain the expression

$$\frac{(\Delta t)^2}{2}(a^2 w_x)_x \tag{30}$$

which strongly resembles the Lax-Wendroff term (16). On the basis of eigenvectors of A expressions (30) and (16) respectively would read

$$\operatorname*{Max}_k |\, a^{(k)} |^2 \cdot I \tag{31}$$

and

$$\begin{bmatrix} |\, a^{(1)} |^2 & & \Phi \\ & \ddots & \\ \Phi & & |\, a^{(p)} |^2 \end{bmatrix}. \tag{32}$$

Evidently (31) is a close but safe norm of (32) in the sense that nonlinear instabilities set on by the vanishing of a diagonal element of (32) (see ref. [5]) should not occur in connection with (31).

Further discussion of condition (26) will be postponed until we have shown its validity for all difference schemes considered in this paper.

3.2. *Two-dimensional case*

We define

$$\Delta = [(\Delta x)^2 + (\Delta y)^2]^{1/2}, \tag{33}$$

$$\lambda_1 = \frac{\Delta t}{\Delta x}, \qquad \lambda_2 = \frac{\Delta t}{\Delta y} \tag{34}$$

and redefine

$$\lambda = (\lambda_1^2 + \lambda_2^2)^{1/2} = \frac{\Delta \cdot \Delta t}{\Delta x\, \Delta y}. \tag{35}$$

Our two-dimensional difference scheme is based on eq. (4), with artificial diffusion terms approximating (in general)

$$\frac{(\Delta x)^2}{2}(\kappa_1 w_x)_x + \frac{(\Delta y)^2}{2}(\kappa_2 w_y)_y. \tag{36}$$

The choice of the diffusion coefficients κ_1 and κ_2 however is constrained by

$$\frac{\kappa_1}{\lambda_1^2} = \frac{\kappa_2}{\lambda_2^2} \equiv \frac{\kappa}{\lambda^2} \tag{37}$$

because for constant κ the two terms in expression (36) should form a Laplacean.[3] Note that κ, like λ, is redefined; in general it is some function of t, x and y and/or w.

The centered difference scheme thus completed becomes

$$\begin{aligned} w_{mn}^{j+1} = w_{mn}^{j} &- \frac{\lambda_1}{2}(f_{m+1\,n}^{j} - f_{m-1\,n}^{j}) - \frac{\lambda_2}{2}(h_{m\,n+1}^{j} - h_{m\,n-1}^{j}) \\ &+ \frac{\lambda_1^2}{2\lambda^2}\{\kappa_{m+\frac{1}{2}\,n}^{j}(w_{m+1\,n}^{j} - w_{mn}^{j}) - \kappa_{m-\frac{1}{2}\,n}^{j}(w_{mn}^{j} - w_{m-1\,n}^{j})\} \\ &+ \frac{\lambda_2^2}{2\lambda^2}\{\kappa_{m\,n+\frac{1}{2}}^{j}(w_{m\,n+1}^{j} - w_{mn}^{j}) - \kappa_{m\,n-\frac{1}{2}}^{j}(w_{mn}^{j} - w_{m\,n-1}^{j})\}, \end{aligned} \tag{38}$$

which also approximates the diffusion equation

$$w_t + f_x + h_y = \frac{\Delta t}{2\lambda^2}\{(\kappa w_x)_x + (\kappa w_y)_y\}. \tag{39}$$

The amplification matrix of scheme (38) is

$$G_{12} = \left[1 - \frac{\kappa}{\lambda^2}\{\lambda_1^2(1 - \cos\alpha) + \lambda_2^2(1 - \cos\beta)\}\right] I - i(\lambda_1 A \sin\alpha + \lambda_2 B \sin\beta). \tag{40}$$

We now define a matrix S and a scalar ψ by

$$S = \frac{\lambda_1 A \sin\alpha + \lambda_2 B \sin\beta}{(\lambda_1^2 \sin^2\alpha + \lambda_2^2 \sin^2\beta)^{1/2}} \tag{41}$$

and

$$\sin\frac{\psi}{2} = \frac{1}{\lambda}\left(\lambda_1^2 \sin^2\frac{\alpha}{2} + \lambda_2^2 \sin^2\frac{\beta}{2}\right)^{1/2}. \tag{42}$$

Insertion of (41) and (42) into (40) yields

$$G_{12} = \{1 - \kappa(1 - \cos\psi)\}I - iS(\lambda_1^2 \sin^2\alpha + \lambda_2^2 \sin^2\beta)^{1/2}. \tag{43}$$

[3] This constraint becomes desirable in the course of the general stability analysis. It appears that the derivation of a practical stability criterion is completely obstructed in the case that κ_1 and κ_1 are not related.

With aid of the Schwarz-type inequality

$$\lambda_1^2 \sin^2 \frac{\alpha}{2} + \lambda_2^2 \sin^2 \frac{\beta}{2} \leqslant \lambda \left(\lambda_1^2 \sin^4 \frac{\alpha}{2} + \lambda_2^2 \sin^4 \frac{\beta}{2}\right)^{1/2} \tag{44}$$

one can easily prove that

$$\lambda \sin \psi \geqslant (\lambda_1^2 \sin^2 \alpha + \lambda_2^2 \sin^2 \beta)^{1/2}. \tag{45}$$

The stability of scheme (38) may hence also be discussed on the basis of the matrix

$$G_{12}^* = \{1 - \kappa(1 - \cos \psi)\}I - i\lambda S \sin \psi, \tag{46}$$

as in any case

$$g_{12}^* \geqslant g_{12} \, . \tag{47}$$

Now G_{12}^* has the same appearance as G_{11} in eq. (22). If the eigenvalues of S are called $s^{(k)}(k = 1,..., p)$ and

$$s \equiv \operatorname*{Max}_k | s^{(k)} | \tag{48}$$

then the stability condition appropriate to G_{12}^* evidently becomes

$$\lambda^2 s^2 \leqslant \kappa \leqslant 1. \tag{49}$$

Though in general the eigenvalues of S cannot simply be expressed in terms of the eigenvalues of A and B (because these matrices do not commute) they may be computed with some difficulty in the particular case that (4) represents the Eulerian equations of plane ICF. Following Richtmyer and Morton closely (ref. [4], Section 13.4) we call

$$\cos \vartheta = \frac{\lambda_1 \sin \alpha}{(\lambda_1^2 \sin^2 \alpha + \lambda_2^2 \sin^2 \beta)^{1/2}} \tag{50}$$

and subsequently obtain

$$\begin{pmatrix} s^{(1)} \\ s^{(2)} \\ s^{(3)} \\ s^{(4)} \end{pmatrix} = \begin{pmatrix} u \cos \vartheta + v \sin \vartheta \\ u \cos \vartheta + v \sin \vartheta \\ u \cos \vartheta + v \sin \vartheta + c \\ u \cos \vartheta + v \sin \vartheta - c \end{pmatrix}, \tag{51}$$

where u, v and c have theit usual meaning of x-, y- and sound velocity.

The $s^{(k)}$ depend on ϑ and so does s, which is inconvenient. We therefore introduce

$$s' \equiv \operatorname*{Max}_\vartheta s(\vartheta) = (u^2 + v^2)^{1/2} + c \tag{52}$$

so that always

$$s \leqslant s'. \tag{53}$$

Replacing s by s' in condition (49) hence yields a safe stability criterion

$$\lambda^2 s'^2 \leqslant \kappa \leqslant 1. \tag{54}$$

Note that $\lambda s'$ is the Courant number for plane Eulerian flow; correspondingly we shall write (54) as

$$(\sigma^j_{mn})^2 \leqslant \kappa^j_{mn} \leqslant 1, \tag{55}$$

the perfect two-dimensional analogon of condition (26). Without proof we state that its validity may even be extended to the three-dimensional case, provided only that the definitions (34), (35) and (37) are properly extrapolated.

Actually condition (55) is too restrictive, because in general the *equal-signs* in (47) and (53) do not operate for the same set of values (α, β). We have not attempted to derive a milder form (depending on u and v in a more intricate manner), because we saw neither the way nor the need to do such.

4. Two-Step Schemes

For the sake of completeness we shall now discuss two-step methods incorporating scheme (19) or (38) as the first step. The aim of the second step is to achieve second-order accuracy; this is accomplished by the implicit introduction of w_{tt}. We may fit all possible two-step schemes into a one-parameter family (*viz.* of r); its one-dimensional version is

$$\left.\begin{aligned} w^*(t + \Delta t) &= w(t) - \Delta t\, f_x(t) + \frac{(\Delta x)^2}{2}\{\kappa w_x(t)\}_x \\ w(t + r\,\Delta t) &= w(t) - r\,\Delta t \left\{\left(1 - \frac{r}{2}\right) f_x(t) + \frac{r}{2} f_x^*(t + \Delta t)\right\}, \end{aligned}\right\} \tag{56}$$

spatial discretization ignored. An asterisk denotes intermediate values at $t + \Delta t$ having only first-order accuracy. It is easily verified that the second step in fact yields

$$w(t + r\,\Delta t) = w(t) - r\,\Delta t\, A w_x + \frac{(r\,\Delta t)^2}{2}(A^2 w_x)_x + O(\Delta t^3) \tag{57}$$

or, in words, values of w accurate to the second order, at the instant $t + r\Delta t$. With a proper choice for κ the two-step scheme may be stable within the range of the

CFL condition. The latter now concerns only the *final* time step and is twice as weak as for the one-step scheme, namely

$$\frac{r\,\Delta t}{\Delta x}\,a \leqslant 2, \tag{58}$$

because after the first step the domain of dependence connected with the difference equations is effectively doubled.

All two-step methods in use at the present start from Lax's scheme (29), because of its simplicity. In the second step r is most often substituted by 2 (as originally proposed by Richtmyer [6]) or by 1 (a choice attributed to Wendroff), though schemes with different values of r have been investigated by Gourlay and Morris [7]. However, in practical applications there is no apparent reason to go beyond the range

$$1 \leqslant r \leqslant 2. \tag{59}$$

Apart from the final weight of the diffusion term ($\sim r^2$), all two-step schemes given by (56) have the same amplification matrix. We shall therefore restrict the discussion to the scheme we consider most elegant, namely the one with $r = 2$. This particular value of r makes the CFL conditions for the two-step scheme and the constituent one-step scheme coincide. As one may anticipate, the stability criteria for these schemes are also identical, a proof of which is given below.

The respective amplification matrices G_{21} and G_{11} of the two-step and one-step schemes considered are related by

$$G_{21} = I - 2i\lambda A G_{11} \sin \alpha \tag{60}$$

and their conjugates by

$$\bar{G}_{21} = I + 2i\lambda A \bar{G}_{11} \sin \alpha. \tag{61}$$

From this it follows that

$$I - G_{21}\bar{G}_{21} = 2i\lambda A(G_{11} - \bar{G}_{11}) \sin \alpha - 4\lambda^2 A^2 G_{11}\bar{G}_{11} \sin^2 \alpha. \tag{62}$$

According to (22) we have

$$G_{11} - \bar{G}_{11} = -2i\lambda A \sin \alpha \tag{63}$$

yielding

$$I - G_{21}\bar{G}_{21} = 4\lambda^2 A^2 (I - G_{11}\bar{G}_{11}) \sin^2 \alpha. \tag{64}$$

As the eigenvalues of $G_{21}\bar{G}_{21}$ and $G_{11}\bar{G}_{11}$ are $|\, g_{21}^{(k)} \,|^2$ and $|\, g_{11}^{(k)} \,|^2$ respectively, eq. (64) already implies that stability of the one-step scheme is necessary and sufficient for stability of the two-step scheme. Hence condition (26) is applicable once more.

The two-dimensional two-step scheme becomes, with $r = 2$,

$$\left.\begin{aligned} w^*(t + \Delta t) &= w(t) - \Delta t\{f_x(t) + h_y(t)\} + \frac{(\Delta x)^2(\Delta y)^2}{2\Delta^2}\{[\kappa w_x(t)]_x + [\kappa w_y(t)]_y\} \\ w(t + 2\,\Delta t) &= w(t) - 2\Delta t\{f_x^*(t + \Delta t) + h_y^*(t + \Delta t)\}, \end{aligned}\right\} \tag{65}$$

and has the amplification matrix

$$G_{22} = I - 2i(\lambda_1 A \sin\alpha + \lambda_2 B \sin\beta)\, G_{12}\,. \tag{66}$$

The appropriate stability condition is (55), as follows from an argument similar to the one given for the one-dimensional case. As for the three-dimensional case, we refer to our statement following condition (55).

Though equally subject to nonlinear instabilities, two-step methods are usually preferred to the one-step Lax–Wendroff method for reasons of computational economy. Yet one should not forget that a two-step scheme still involves artificial diffusion, albeit of order $O(\Delta t^3)$ and perhaps locally minimized. *The Lax–Wendroff scheme remains the only second-order technique that does not ultimately destroy contact discontinuities in a Lagrangean mesh.*

5. Examples

We shall illustrate the implications of conditions (26) and (55) on the basis of a set of simple expressions for the diffusion coefficient (the subscript-n is optional and therefore parenthesized)

$$\kappa^j_{m(n)} = \varphi^j \sigma^j \left(\frac{\sigma^j_{m(n)}}{\sigma^j}\right)^N, \qquad N = 0, 1 \text{ or } 2. \tag{67}$$

Here φ is a position-independent parameter which may be adjusted in order to meet special damping requirements. The stability conditions (26) and (55) can now be expressed in terms of φ:

$$\sigma^j \leqslant \varphi^j \leqslant \frac{1}{\sigma^j}\,. \tag{68}$$

There are three values of φ that deserve special attention, *viz.* the interval bounds in (68), and 1. The reader should realize that the diffusion term

$$\frac{\Delta t}{2\lambda^2}\nabla\cdot(\kappa\,\nabla w) \tag{69}$$

occurring in eqs. (21) and (39) will be proportional to Δt if $\varphi \equiv \sigma$, independent of Δt if $\varphi \equiv 1$, and inversely proportional to Δt if $\varphi \equiv 1/\sigma$. We will briefly go over the different values of N.

$N = 0$; *diffusion term not adapted to local circumstances.*

As mentioned before, the case $\varphi \equiv 1/\sigma$, or $\kappa \equiv 1$, corresponds to Lax's scheme (for an arbitrary number of dimensions). It is well-known that, when employing this scheme, one should always take the largest value of Δt permitted by the CFL condition in order to avert excessive smearing. An important improvement would be achieved by the use of $\varphi \equiv 1$ or σ, hence $\kappa \equiv \sigma$ or σ^2. However, for any $\kappa \not\equiv 1$ the simple triangular structure of Lax's scheme is lost; therefore one may as well choose $N = 1$ or 2.

$N = 1$; *diffusion term proportional to the local maximum characteristic speed*

This is exactly the form adopted by Rusanov [2], and needs no further comment. Note that for $\varphi \equiv 1$ the one-dimensional diffusion term (17) approximates

$$\frac{\lambda}{2}(aw_x)_x(\Delta x)^2, \tag{70}$$

an expression which also appears in the method of Godunov [8].

$N = 2$; *diffusion term proportional to the square of the local maximum characteristic speed*

To the author's knowledge this quadratic dependence has found no application so far; yet it clearly yields the most accurate first-order methods that are possible. The accuracy finds expression in a high resolution of spatial detail. With $\varphi \equiv \sigma$, the diffusion at any point equals *the minimum needed for local stability*; this minimum closely bounds the Lax-Wendroff term that would give second-order accuracy (cf. Section 3.1). Even in the case of minimum diffusion nonlinear instabilities are not expected.

To compare the merits of the different diffusion coefficients represented by (67), we have carried out, with aid of scheme (19), some numerical integrations of the single nonlinear hyperbolic equation

$$w_t + (\tfrac{1}{2}w^2)_x = 0, \tag{71}$$

starting from a discontinuous set of initial values. Some trivial suppositions concerning shocks were confirmed: the shock width decreases with increasing N or decreasing φ, but simultaneously the overshoot grows. The value $\varphi \equiv 1$ appears to be optimal in the sense of generating narrow shocks with little overshoot; for $N = 1$ this result was derived by Godunov [8] and experimentally found by Emery [9]. By choosing Δt close to the CFL stability limit, both overshoot and shock width may be reduced; this nonlinear effect has also been observed by Rubin and Burstein [10] for two-step schemes.

REFERENCES

1. P. D. LAX, *Comm. Pure Appl. Math.* **7**, 159 (1954).
2. V. V. RUSANOV, *Zhur. Vych. i Mat. Fiz.* **1**, 267 (1961); also NRC of Canada Technical Translation 1027 (1962).
3. P. D. LAX and B. WENDROFF, *Comm. Pure Appl. Math.* **13**, 217 (1960).
4. R. D. RICHTMYER and K. W. MORTON, "Difference Methods for Initial-Value Problems," 2nd edition. Interscience, New York, 1967.
5. S. Z. BURSTEIN, *J. Computational Phys.* **1**, 198 (1966).
6. R. D. RICHTMYER, A survey of difference methods for non-steady fluid dynamics. NCAR Technical Note 63-2 (1962).
7. A. R. GOURLAY and J. Ll. MORRIS, *Math. Comp.* **22**, 28 (1968).
8. S. K. GODUNOV, *Mat. Sb.* **47**, 271 (1959), also Cornell Aeronautical Lab. translation.
9. A. F. EMERY, *J. Computational Phys.* **2**, 306 (1968).
10. E. L. RUBIN and S. Z. BURSTEIN, *J. Computational Phys.* **2**, 178 (1967).